Computer Science Department

TECHNICAL REPORT

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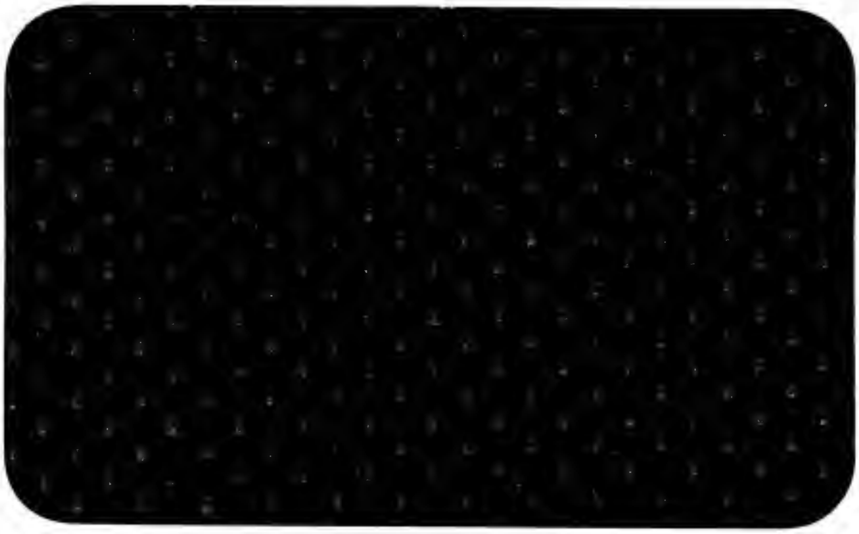
Technical Report #233
August, 1986



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† This work was supported the National Science Foundation under Grant NSF DCR-8405506, and by the U.S. Department of Energy, under Contract DE-AC02-76ER03077-V, at the Courant Mathematics and Computing Laboratory.

‡ This report was prepared for the proceedings of the second GAMM-SEMINAR on Efficient Numerical Methods in Continuum Mechanics, held in Kiel, Federal Republic of Germany, January 17-19, 1986.

AN EXTENSION THEOREM FOR FINITE ELEMENT SPACES WITH THREE APPLICATIONS

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SUMMARY

In this paper, an extension theorem, similar to well-known results for Sobolev spaces, is established for general conforming finite element spaces. Results of this type provide central tools for the theory of iterative substructuring (domain decomposition) and capacitance matrix methods. The main theorem has previously been established, using other techniques, for certain Lagrangian finite elements in the plane. Several applications are discussed in which previous results on iterative substructuring and capacitance matrices are extended to three dimensions, non-Lagrangian finite elements and higher order elliptic problems.

1. INTRODUCTION

Recently there has been a considerable interest in the development and study of iterative substructuring methods for elliptic finite element problems; see Bjørstad and Widlund [6 , 7], Bramble, Pasciak and Schatz [8 , 9], Chan [11], Chan and Reasco [12], Dihn, Glowinski and Périaux [16] , Dryja [17,18], Dryja and Proskurowski [19,20], Golub and Mayers [21], Keyes and Gropp [22] and Widlund [29]. Early work in this area is described in Concus, Golub and O'Leary [15], while the paper by Dryja [17] appears to be the first in which an optimal iterative method is described and analyzed. Much of this work has been for finite difference approximations of second order elliptic problems, often on relatively special plane regions. It is the purpose of this paper to go beyond these cases to more difficult and important finite element problems.

When using these algorithms, also known as domain decomposition methods, the discretized elliptic problem is partitioned into subproblems which correspond to non-overlapping subsets of the region. The subproblems are then solved separately, and repeatedly, while the interaction between the subregions is handled by a conjugate gradient or other suitable iterative method. These iterative methods provide interesting alternatives to the standard industrial finite element practice in which not only the stiffness matrices corresponding to the finite element models for the substructures but also the

matrices which represent the interaction between the different parts are fully assembled and factored into their Choleski triangular factors; see Bell, Hatlestad, Hausteen and Ar ldsen [2] and Przemieniecki [25]. For a discussion of different algorithms etc. and a general finite element framework, see Bjørstad and Widlund [7].

The fastest among these methods offers considerable advantages even on sequential computers. They also show particular promise for parallel computing since the last stage of a block Gaussian factorization is likely to lend itself less well to parallel architectures than the earlier stages where work can be carried out for the different substructres without any need of communication between them. Work on the actual parallel inplementation of these methods is beginning; see Keyes and Gropp [22] for results on an Intel Hypercube. Similarly, variants of methods for general sparse linear systems of algebraic equations, such as nested dissection algorithms, could be developed in which the factorization would be stopped at a suitable stage. The remaining reduced system, corresponding to the variables not eliminated, would then be solved by an iterative method.

The rate of convergence of iterative methods of this kind can of course be studied by conducting systematic numerical experiments. For general sparse or band matrices it is unlikely that strong general convergence results for powerful preconditioners will be developed for general systems of equations. When a problem arises from the discretization of an elliptic problem additional tools from mathematical analysis are available and the possibility of a systematic development of good preconditioners can become a reality. Thus for an elliptic problem with constant coefficients on a uniform mesh and a region which is a union of a few rectangles, Fourier series can be used; see e.g. section 4 of Bjørstad and Widlund [7]. Similar techniques are used in Bjørstad [4,5] to study a biharmonic problem and in Chan [10] to develop and analyze new fast methods for queuing networks. However, a systematic theory for elliptic problems on general regions requires the development of finite element analogues of certain regularity results for inhomogeneous elliptic boundary value problems, see Bjørstad and Widlund [6, 7] and Bramble, Pasciak and Schatz [8, 9]. The technical aspects of this work is complicated by the fact that the boundaries of the subregions necessarily have corners.

As in the continuous case, we can reduce an elliptic problem with nonhomogeneous boundary data to one which is homogeneous by using an extension theorem; see e.g. Stein [21]. In recent papers, see Bjørstad and Widlund [7] and Bramble, Pasciak and Schatz [8], an extension theorem has been developed for conforming Lagrangian finite elements in the plane. This proof, which essentially has been known at least since 1980, uses a variant of the Bramble-Hilbert lemma; see Ciarlet [13], and a regularity theorem for elliptic differential equations. It appears difficult to extend this proof to non-Lagrangian finite elements, three dimensions and higher order equations since it relies on the boundedness, in a

Sobolev space, of the finite element interpolation operator. The choice of this Sobolev space is limited by the lack of regularity of the solutions of elliptic problems on regions with corners. In this paper, we therefore proceed differently, inspired by an idea used by Astrakhantsev [1] to establish that a capacitance matrix method for a rather special variational difference scheme is optimal for the Neumann boundary condition.

We note that the extension theorems are related to questions concerning the approximation order of finite element spaces for functions which are not sufficiently smooth. These issues were discussed early by Strang [28] for cases without boundaries. This work was extended to some extent by Clément [14] and later quite systematically by Bernardi [3]. While Bernardi carefully includes boundary conditions in her work, her work does not include Hermitian finite elements and it is therefore not directly applicable to all the problems at hand.

In section 3, we discuss three applications. We thus extend Astrakhantsev's result on capacitance matrix methods for Neumann problems to a much more general family of finite elements. While, at least in the West, iterative substructuring methods are attracting much more attention than capacitance matrix algorithms the two families of methods are nevertheless closely related. We also give two new results on domain decomposition in this last section.

2. THE EXTENSION THEOREM

Let Ω be an open, bounded region in R^d, $d \geq 2$, with a piecewise smooth, uniformly Lipschitz continuous boundary Γ. It is well known, cf. e.g. Stein [27], Chapter VI, that there exists a linear operator $\mathcal{E}$ extending functions on Ω to functions on R^d with the properties

i) $\mathcal{E}u|_{\Omega} = u$,

i.e. $\mathcal{E}$ is an extension operator,

$$\text{ii)} \quad ||\mathcal{E}u||_{H^m(R^d)} \leq C(\Omega)||u||_{H^m(\Omega)} , \tag{2.1}$$

i.e. $\mathcal{E}$ maps $H^m(\Omega)$ continuously into $H^m(R^d)$. m is a nonnegative integer.

The constant $C(\Omega)$ depends on d, m and the Lipschitz constant of the region only. The Sobolev space $H^m(\Omega)$ is the subspace of $L_2(\Omega)$ such that

$$||u||_{H^m(\Omega)} = \left(\int_{\Omega} \sum_{|\alpha|\leq m} |(\tfrac{\partial}{\partial x})^{\alpha} u|^2 \, dx \right)^{1/2} < \infty ,$$

with $(\frac{\partial}{\partial x})^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_d})^{\alpha_d}$ and $|\alpha| = \sum \alpha_i$.

We will develop an analogue of this extension theorem for finite element spaces. Let τ^h be a triangulation of Ω into elements. For convenience we assume that each element K is a simplex with sufficiently smooth faces; our theory could equally well be developed for quadrilaterals. The simplices can be curved. As usual we assume that the simplices are properly joined which means that the intersection of any two of them is empty or an entire face of dimension d-1 or less. By h_K we denote the diameter of K and by ρ_K the diameter of the largest ball contained in the element. Following Bernardi [3], we assume that each element is the image of a straight d-simplex $\hat{K}$ under a C^1 map F_K of the form

$$F_K = \tilde{F}_K + \Phi_K$$

where $\tilde{F}_K$ is an invertible affine mapping

$$\tilde{F}_K : \hat{x} \to \tilde{B}_K \hat{x} + \tilde{b}_K$$

and Φ_K satisfies

$$c_K = \sup_{\hat{x} \in \hat{K}} ||D\Phi_K(\hat{x}) \hat{B}_K^{-1}|| \leq c < 1$$

with c uniformly bounded away from 1. We note that $\tilde{K} = \tilde{F}_K(\hat{K})$ is a straight d-simplex and that we can arrange it so that K and $\tilde{K}$ have the same vertices, i.e. $\tilde{K}$ is a straight approximation of K. Isoparametric elements as well as those based on an exact triangulation, cf. Bernardi [3], Lenoir [23] and Scott [26], can easily be accommodated in this framework. The triangulation is assumed to be regular, i.e. $\tilde{h}_K/\tilde{\rho}_K \leq \sigma$, where σ is uniformly bounded for all h. These assumptions exclude degenerate, very flat elements and also lead to a bound on the number of simplices that have a particular vertex in common, but still allow us selectively to refine the triangulation to improve the accuracy where the solution is less smooth. In practice a simplex will be straight, unless at least two of its vertices fall on the boundary or an interface between subregions modeled separately e.g. because of different material properties. We will not discuss the effects of approximating Ω by a union of straight of isoparametric elements but always assume that the triangularions of Ω are exact.

We will consider general, conforming finite elements, i.e. the approximating space $V^h \subset V$ where V is the linear space appropriate for the elliptic variational problem to be considered; see Ciarlet [13]. A finite element is defined on the element level by a triple (K, P_K, Σ_K) where K is the simplex and P_K a space of functions. The dimension of P_K is bounded uniformly. Normally P_K is the image under the map F_K of a space of polynomials. We note that these functions can be vectored values. No essential difficulties are introduced by such an assumption nor by considering curved rather than straight simplices. Σ_K is a set of linear functionals defined on smooth functions and on P_K such that the related interpolation problem on K is uniquely solvable in P_K.

From the set $\bigcup_{K \in \tau_h} \Sigma_K$ a maximal system of linearly inde-

pendent linear functionals on $C^\infty(\bar{\Omega})$ is extracted. Basis functions for the finite element space V^h can then be constructed as e.g. in Ciarlet [13], creating $\{\phi_i\}_{i=1}^{N_h}$ and $\{\mu_i\}_{i=1}^{N_h}$ which are a dual pair of bases for the space V^h and the space of linear functionals, respectively i.e.

$$\mu_i(\phi_j) = \delta_{ij}, \quad 1 \le i \le N_h, \quad 1 \le j \le N_k .$$

For $1 \le i \le N_h$, we define

$$\Delta_i = \bigcup_{K\in\tau^h} \{K;\ \text{supp}\ \mu_i \text{ overlaps } K\} .$$

Under the assumptions given above one can show straightforwardly that the number of elements that form Δ_i is uniformly bounded and that the diameters of any elements $\bar{K}$ and K', contained in the same Δ_i satisfy

$$\tilde{h}_K \le C\tilde{h}_{K'},$$

where C depends on $c = \max c_K$ and $\sigma = \max \tilde{h}_K/\rho_K$ only. The support of ϕ_i is contained in Δ_i.

In our proof we assume, following Strang [28], that the basis functions $\phi_i(x)$ are uniform of order m, i.e. there exists constants c_s, independent of h and i, such that

$$\max_{\substack{x\in\Delta_i \\ |\alpha|=s}} |(\tfrac{\partial}{\partial x})^\alpha \phi_i(x)| \le c_s\, h^{d_i-s}, \quad s \le m . \tag{2.2}$$

We note that d_i often is the degree of a derivative associated with the degree of freedom in question.

Assuming that $C\Omega$, the complement of Ω, has been triangulated in an equally benign way and that $V^h(\Omega)$ has been extended to $V^h(R^d) \subset H^m(R^d)$, we are ready to describe how to find an extension $\mathcal{E}^h u_h \in V^h(R^d)$ of a given element $u_h \in V^h(\Omega)$ such that

$$||\mathcal{E}^h u_h||_{H^m(R^d)} \le C||u_h||_{H^m(\Omega)} . \tag{2.3}$$

In a first step, we use the original extension theorem and the fact that the space is conforming, i.e. $u_h \in H^m(\Omega)$, to find $\mathcal{E}u_h \in H^m(R^d)$ such that

$$||\mathcal{E}u_h||_{H^m(R^d)} \le C(\Omega)||u_h||_{H^m(\Omega)} . \tag{2.4}$$

However $\mathcal{E}u_h \notin V^h(R^d)$.

In a second step, we follow Strang [25] and smooth $\mathcal{E}u^h$ and interpolate to find an element $w_h \in V^h(R^d)$ such that

$$||w_h||_{H^m(R^d)} \le C||\mathcal{E}u_h||_{H^m(R^d)} \le C(\Omega)C\,||u_h||_{H^m(\Omega)} . \tag{2.5}$$

However $w_h|_\Omega \ne u_h$ and w_h is therefore not an extension of u_h. We note that Strang [28] shows that

$$||w_h - \&u_h||^2_{L^2(K)} \leq Ch_K^{2m} \sum_{K'}^{fin} ||w_h - \&u_h||^2_{H^m(K')} \quad . \qquad (2.6)$$

where the sum is over a fixed number of neighboring elements.

In a third step, we patch w_h and u_h together to create $\&^h u_h$. We interpolate using all the original parameters obtained from u_h in $\bar{\Omega}$ and in $C\bar{\Omega}$, the complement of $\bar{\Omega}$, those associated with w_h. The resulting function $\&^h u_h \in V^h(R^d)$. For any element $K \subset \Omega$ the original values of u_h are recovered. We also note that similarly, in $C\Omega$, $\&^h u_h$ differs from w_h only on elements which have at least one vertex on Γ. There remains to establish (2.3).

By the triangle inequality

$$||\&^h u_h||_{H^m(C\Omega)} \leq ||\&^h u_h - w_h||_{H^m(C\Omega)} + ||w_h||_{H^m(C\Omega)} \quad .$$

By (2.1) and (2.5), $||w_h||_{H^m(C\Omega)}$ can be estimated by $||u_h||_{H^m(\Omega)}$.

Consider an element $K' \subset C\Omega$, with at least one vertex on Γ for which $\&^h u_h - w_h$ is not identically zero. The parameters associated with the local interpolation problem are equal to zero except for those shared with at least one element $K \subset \Omega$. By using (2.2) one can establish that

$$||\&^h u_h - w_h||^2_{H^m(K')} \leq ch_K^{-2m} ||\&^h u_h - w_h||^2_{L^2(K')} \quad . \qquad (2.7)$$

Under the regularity assumptions introduced above and (2.2), we can estimate this L_2 norm by $h_K^{d/2} \times$ the ℓ_2 norm of the vector of parameters $h^{d_i} \mu_i(\&^h u_h - w_h)$. When considering the corresponding norms of $\&^h u_h - w_h$ on the neighboring element $K \subset \Omega$, we see that

$$||\&^h u_h - w_h||_{L^2(K')} \leq C||\&^h u_h - w_h||_{L^2(K)} = C||\&u_h - w_h||_{L^2(K)} .$$

The crucial observation is that this inequality holds whatever the values of the additional parameters associated with K happen to be.

The proof is concluded by adding over all the relevant elements of $C\Omega$ and using inequalities (2.6) and (2.7). An exact analogue of the extension theorem given in the beginning of this section has thus been established for conforming finite element spaces defined on sufficiently regular triangulations. We note that $\&^h$ is a linear operator and if desired we can modify $\&^h u_h$ so that it vanishes at a fixed distance outside Γ.

3. APPLICATIONS

In this section, we will briefly discuss the application of the extension theorem to capacitance matrix and iterative substructuring methods. We note that the main idea behind the

proof of our first result is due to Astrakhantsev [1].

There are occasions when a linear system of equations can be imbedded in a larger system which is easier to solve. For the expanded system an efficient preconditioner might be available e.g. if the geometry of the corresponding region is rectangular and fast Poisson solvers can be used; see Proskurowski and Widlund [24] and the references therein.

Here we will only consider the solution of Neumann problems for selfadjoint elliptic problems by capacitance matrix methods. The region Ω is imbedded in Λ, a rectangle or some other convenient, simple region. The boundaries of Ω and Λ do not intersect and the two regions are triangulated as in section 2. A symmetric bilinear form $a_\Omega(u,v)$ is associated with the original elliptic problem. To avoid nonessential details, we assume that the problem is strictly elliptic in the sense that

$$c\|u\|^2_{H^m(\Omega)} \le a_\Omega(u,u) .$$

We also assume that this bilinear form can be extended to a similarly well behaved form $a_\Lambda(u,v)$. We note that we have considerable freedom in choosing a boundary condition on the boundary of Λ.

The space $H^m(\Lambda)$, or an appropriate subspace of functions satisfying these boundary conditions, is approximated by $\tilde{V}^h(\Lambda)$, a conforming finite element space satisfying the conditions of section 2. The restriction $V^h(\Omega)$ of this space to Ω is used to solve the Neumann problem on Ω. We note that all the parameters associated with $V^h(\Omega)$ are determined when the discrete Neumann problem is solved. Denote by A the resulting positive definite, symmetric stiffness matrix and by n_h the order. The matrix corresponding to the problem on the larger region Λ is denoted by B and its order is N_h.

One can think of the capacitance matrix method as a preconditioned conjugate gradient method in which the trivially expanded system

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

is solved using B as a preconditioner. As always the convergence of the method is determined by the spectrum of

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \phi = \lambda B \phi$$

or equivalently by the stationary values of the Rayleigh quotient

$$\frac{x_1^T A x_1}{x^T B x} .$$

Let $\lambda_1 \ge \lambda_2 \ge \dots$ be the eigenvalues. We note that

$\lambda_{n_h+1} = \lambda_{n_h+2} = \ldots = \lambda_{N_h} = 0$ and that $\lambda_1 \leq 1$ since the strain energy on Λ is larger than that on the smaller region Ω. The null space of dimension $N_h - n_h$ does not affect the algorithm, cf. Astrakhantsev [1] or Proskurowski and Widlund [24]. To prove that the rate of convergence is independent of the mesh size we must establish a good lower bound on λ_{n_h}. Such a bound is obtained by the Courant-Fischer theorem. The $N_h - n_h$ linear constraints used are obtained by using the extension theorem. The value of each degree of freedom associated with $\Lambda \setminus \Omega$ is, by the linearity of $\mathcal{E}^h$, a linear combination of those associated with $V^h(\Omega)$.

The extension theorem therefore provides a bound of the strain energy on Λ in terms of that on Ω and the proof is completed. We note that numerical experiments reported in Proskurowski and Widlund [ ] show that this bound, and thus the constant in the extension theorem, teriorates if triangles become very thin.

We next turn to a discussion of an optimal iterative substructuring method; cf. Bjørstad and Widlund [7] for a detailed discussion of this and similar methods. We begin by discussing Lagrangian finite elements, extending our previous results to dimensions higher than two. Since Lagrangian elements do not belong to H^2, we confine ourselves to second order systems. To simplify the notations, we also concentrate on the case where the original region Ω is the union of two regions Ω_1 and Ω_2 and Γ_3, the intersection of the closures of Ω_1 and Ω_2. The boundaries of Ω_1, Ω_2 and Ω are $\bar{\Gamma}_1 \cup \bar{\Gamma}_3$, $\bar{\Gamma}_2 \cup \bar{\Gamma}_3$ and $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$ respectively. We will work with a Dirichlet condition on the boundary of Ω.

The variational formulation is then

$$a_\Omega(u_h, v_h) = f(v_h), \quad \forall\, v_h \in V_0^h(\Omega) \quad . \tag{3.1}$$

$$u_h \in V^h(\Omega), \; u_h \text{ given on } \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \; ,$$

where $V_0^h(\Omega)$ is the subspace of functions in $V^h(\Omega)$ which vanish on $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$. We again assume that $a_\Omega(u,v)$ is uniformly elliptic and selfadjoint.

In the proof, we need to tend an element of $V_0^h(\Omega_1, \Gamma_1)$, the subspace of $V^h(\Omega_1)$ of functions vanishing on Γ_1, to an element of $V_0^h(\Omega)$. We can accomplish this by extending the given element by zero in $C\Omega$ and then use the construction of section 2 to obtain values in Ω_2.

The linear system of equations corresponding to (3.1) has the form

$$Kx = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3.2}$$

where K is positive definite and symmetric. The matrix K_{11} represents the couplings between pairs of degrees of freedom in Ω_1, K_{13} couplings between pairs belonging to Ω_1 and Γ_3 respectively etc.

We make the assumptions that the discrete problems on the subregions, with some appropriate boundary condition added on Γ_3, can be solved exactly. We can therefore set b_1 and b_2 equal to zero. By block Gaussian elimination, the problem (3.2) is reduced to

$$Sx_3 = (K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23}) x_3 = b_3 \ . \qquad (3.3)$$

This system is preconditioned by $S^{(1)}$, a matrix related to a problem on Ω_1 as follows. By using a natural boundary condition on Γ_3 as the only nonhomogeneous data and a zero Dirichlet condition on Γ_1, we obtain

$$\begin{pmatrix} K_{11} & K_{13} \\ K_{13}^T & K_{33}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_3 \end{pmatrix} , \qquad (3.4)$$

where the elements of $K_{33}^{(1)}$ have the form $a_{\Omega_1}(\phi_i, \phi_j)$, with ϕ_i, ϕ_j basis functions associated with degrees of freedom on Γ_3.

The problem reduces to

$$S^{(1)} x_3 = (K_{33}^{(1)} - K_{13}^T K_{11}^{-1} K_{13}) x_3 = c_3$$

Since $S = S^{(1)} + S^{(2)}$, where $S^{(2)}$ is constructed in the same way as $S^{(1)}$, it is not surprising that $S^{(1)}$ is an excellent preconditioner of S.

To provide upper and lower bounds for the associated generalized eigenvalue problem, we consider the Rayleigh quotient

$$\frac{x_3^T \, S^{(1)} x_3}{x_3^T \, S \, x_3} \ .$$

We call a solution of (3.4) a discrete harmonic and we similarly say that the restrictions to Ω_1 and Ω_2 of the solution corresponding to (3.2) with zero b_1 and b_2 is a discrete harmonic function on Ω_1 as well as on Ω_2. The numerator of the Rayleigh quotient represents the strain energy of a discrete harmonic function on Ω_1 while the denominator also contains the strain energy of its discrete harmonic extension to Ω_2. An upper bound is therefore trivially obtained. We need to estimate $x_3{}^T S^{(2)} x_3$ in terms of $x_3{}^T S^{(1)} x_3$ to obtain a lower bound. We note that

$$x_3^T S^{(2)} x_3 = a_{\Omega_2}(u_h, u_h)$$

for some $u_h \in V_0^h(\Omega_2, \Gamma_2)$ which also satisfies

$$a_{\Omega_2}(u_h, v_h) = 0 \ , \qquad \forall \ v_h \in V_0^h(\Omega_2) \ .$$

Therefore

$$a_{\Omega_2}(u_h+v_h,\ u_h+v_h) = a_{\Omega_2}(u_h,u_h) + a_{\Omega_2}(v_h,v_h)\ ,\quad \forall\, v_h \in V_0^h(\Omega_2),$$

i.e. u_h has the least strain energy of any element in $V_0^h(\Omega_2,\Gamma_2)$ with the given boundary values on Γ_3. In view of this the extension theorem, as modified in this section, can be used to complete the argument.

We conclude with some comments on a non-Lagrangian case. To be specific, we consider the Dirichlet problem for the biharmonic equation approximated by the 18 degrees of freedom, reduced quintic element discussed in Ciarlet [13]. The interpolation conditions are associated with the values of the function and all its first and second derivatives at the vertices of the triangles. Here we principally want to make the point that there are two different ways of obtaining an $H^2(\Omega)$ extension from the subregion Ω_1 to Ω_2 and therefore two domain decomposition algorithms. For simplicity, we assume that Γ_3 is straight.

In the first variant all the degrees of freedom associated with Γ_3 are shared by the finite element functions on Ω_1 and Ω_2. The proof of the optimality of the domain decomposition algorithm carries over directly from the case discussed above. In the iteration, approximate values of six parameters per vertex on Γ_3 are obtained. In the second variant, we keep the values of $\dfrac{\partial^2 u_h}{\partial n^2}$ free when extending the discrete biharmonic function from Ω_1 to Ω_2. Since this corresponds to the removal of constraints we have no further problem establishing the necessary bound. From an algorithmic point of view there is a possible benefit. A detailed examination of the linear algebra involved reveals that we only obtain five nontrivial residuals per vertex on Γ_3, corresponding to the parameters shared between Ω_1 and Ω_2. We note that the values of $\partial^2 u_h/\partial n^2$ at the vertices do not affect the values of u_h and its gradient on Γ_3. A solution obtained by this second variant will thus belong to H^2 but in general it will differ from the standard discrete solution obtained by applying the finite element method on the entire region Ω.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant NSF-DCR-8405506, and by the U. S. Department of Energy, under Contract DE-AC02-76ER03077-V, at the Courant Mathematics and Computing Laboratory.

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